Application of Schapery's Theory of Viscoelastic Fracture to Solid Propellant

S. R. Swanson*
University of Utah, Salt Lake City, Utah

The analysis of time-dependent crack propagation in viscoelastic materials in general, and solid propellants in particular, has been hampered by the difficulty of the mathematical analysis of a cracked viscoelastic material. The viscoelastic solution to the singular line crack has not provided a realistic fracture criterion, whereas various approximations to this problem have compared more favorably with experiments. However, it is difficult to assess the generality of the approximations involved. Recently, Schapery has generalized the Barenblatt model to the viscoelastic case and thus developed a model that appears to be consistent with both theory and experiment. The objective of the present work was to obtain a detailed comparison of this model with laboratory results available in the literature on a PBAN solid propellant. The results of the comparison are extremely good over a wide range of variables. A time-dependent fracture energy is found to result which can be incorporated readily into the theory.

Introduction

It is well known that cracks in viscoelastic solid-propellant rocket motor grains may initiate and propagate under various environmental loadings. The consequences of crack propagation during firing of the propellant grain depend critically on the time required for crack propagation compared with the burning rate, as burning can modify the geometry of the crack or flaw. Slowly propagating or stationary cracks thus may be removed essentially by burning, whereas more rapidly propagating cracks may become more severe as the crack deepens, and pressure may build up in the crack.

Viscoelastic crack propagation has been studied by a number of investigators,[1-10] and the results have been developed concerning the time dependence of the initiation of crack propagation and the subsequent velocity of crack propagation. Although the work cited has been a generalization of the classic elastic crack instability analysis, certain physically or mathematically based approximations necessarily have been made because of the complexity of the viscoelastic stress analysis of the crack geometry.

The exact solution to the problem of a line crack in a linearly viscoelastic material has been presented recently by Graham[11] and subsequently corroborated by Nuismer[12] using the usual thermodynamic power balance for fracture. The result of this solution is that the fracture criterion is given by

$$\sigma_0(t_f) = [2\gamma_c/\pi D_g a_0]^{1/2} \tag{1}$$

which is Eq. (11) by Nuismer.[12] It is seen that this is identical to the classic Griffith solution except that the elastic modulus is replaced by the reciprocal of the glassy creep compliance. A second feature of the preceding solution is that no information is given about crack propagation velocities; the criterion applies only to the initiation of cracking.

As pointed out by Nuismer, the foregoing result is physically unappealing in that it appears to be only an upper bound on the fracture stress. No information is available from this result about time-dependent fracture at lower stress levels or crack velocities. Thus the exact solution to the singular viscoelastic line crack problem does not give a realistic fracture criterion, and as discussed by Nuismer, raises question about the validity of the results obtained by the various approximate theories.

Knauss[13,14] has pointed out, however, that a length parameter not present in the singular line crack problem is necessary to introduce time or velocity effects into the viscoelastic fracture analysis. Thus an approximate solution that incorporates a failure zone length, as developed by Knauss,[5,8] may capture more of the physical features of the real case than does the singular line crack, even though an approximate stress analysis was employed.

A way out of this dilema has been developed recently by Schapery.[15,16] Schapery has generalized the Barenblatt model[17] for elastic fracture to the linear viscoelastic case. In Schapery's model (as in the Barenblatt model), a small "cohesive" zone is assumed to exist at the tip of the crack which exerts tractions on the crack faces. The singularity in stress at the crack tip due to these cohesive forces is equated to the negative of the singularity in stress at the crack tip due to the external applied loads, so that the resulting stress is everywhere finite. Schapery calculates the work done on the cohesive zone by the surrounding linear viscoelastic material and equates this to the fracture energy. Schapery develops the equation for crack velocity as

$$C_v(\tilde{t}_\alpha) = 8\Gamma/K_I^2 \tag{2}$$

where C_v is related to the creep compliance, Γ is surface energy, and K_I is the opening mode stress intensity factor. The term $\tilde{t}_\alpha$ can be viewed as the time required for the crack to traverse the cohesive zone at the crack tip. For a one- or two-term power law representation of C_v, this equation can be solved explicitly for the crack velocity as

$$\dot{a} = \left[\frac{C_I \lambda_n \Pi^n}{\Gamma \sigma_m{}^{2n} I_I{}^{2n} 2^{3+n}}\right]^{1/n} K_I{}^{2[1+(1/n)]} \tag{3}$$

where C_v has been taken as $C_v = C_I t^n$. C_v is related approximately to the creep compliance $D(t)$ by $C_v(t) \approx 4(1-\nu^2)D(t)$. The terms in brackets are constants or material parameters developed by Schapery. If the fracture energy Γ is a constant, the propagation law is of the form

$$\dot{a} = A\ K_I{}^{2[1+(1/n)]} = A\ K_I{}^q \tag{4}$$

Received Aug. 22, 1975; revision received Jan. 5, 1976. This work was supported in part by Hercules Incorporated. Discussions with J. J. Anderson, S. C. Beckwith, and R. A. Schapery were very helpful.
Index category: Solid and Hybrid Rocket Engines.

*Research Associate Professor, Department of Mechanical Engineering; also Consultant, Hercules Incorporated.

and the crack propagation velocity is related to the instantaneous value of K_I raised to a power, which, in turn, is determined by the creep compliance. Schapery raises the possibility that Γ depends on $\dot{a}$. A key result of Schapery's analysis is that the crack propagation fracture energy cannot depend on higher derivatives of the crack motion or histories of the crack motion.

Schapery also has derived a criticality condition for initiation of crack spreading which takes the form

$$\Gamma = (K_I{}^2/8)(t_i) \quad C_v{}^{(2)}(t_i) \tag{5}$$

where

$$C_v{}^{(2)}(t) = \frac{1}{K_I{}^2(t)} \int_0^t C_v(t-\tau) \frac{\mathrm{d}K_I{}^2(\tau)}{\mathrm{d}\tau} \mathrm{d}\tau \tag{6}$$

Schapery also raises the question of the possible history dependence of Γ, and whether Γ for initiation is the same as Γ for propagation.

Schapery's theory is seen to have far-reaching implications and apparently ties together many aspects of viscoelastic crack propagation. It is thus appealing from both a practical and theoretical viewpoint. The physical basis of the theory, the cohesion zone of the Barenblatt model, lies in the area between molecular processes and continuum mechanics, and it is difficult to make firm statements about the applicability of the underlying assumptions. Thus it is considered imperative to check the predictions of Schapery's theory against experimental results.

The objective of the present work is to carry out the comparison of theory and experiment just suggested. To this end, the published experimental results of Francis et al.,[18] Bennett,[19] and Jacobs et al.[20,21] are examined. These results all were obtained on a PBAN solid propellant, and taken together, provide a fairly extensive experimental description of the crack propagation behavior of this propellant.

Comparison with Experiment

The tensile relaxation behavior of the PBAN propellant has been presented by Bennett[19] and is shown in Fig. 1. Time-temperature superposition has been employed by Bennett to extend the range of data as shown. The data are seen to plot essentially as a straight line in log-log coordinates, indicating that the power law

$$E_{\mathrm{rel}}(t) = E_I t^{\beta} \tag{7}$$

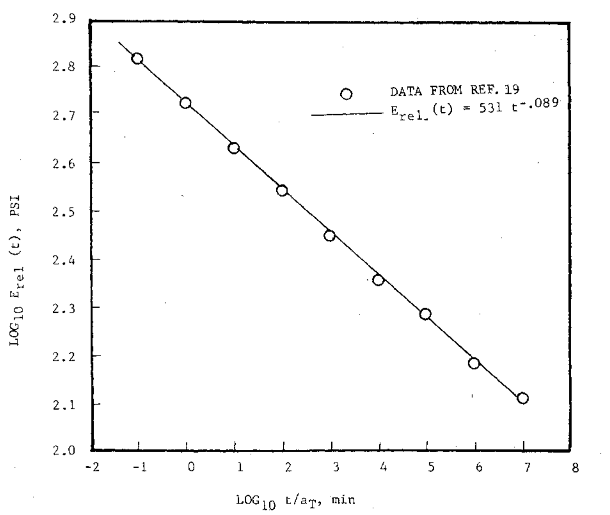

Fig. 1 Stress relaxation modulus for PBAN solid propellant from Ref. 19.

where t is in minutes, with $E_I = 531$ psi and $\beta = -0.089$, can be used to describe the relaxation behavior adequately.

The crack propagation velocity has been measured by Francis et al.[18] in constant load experiments. The time-temperature shifted master curve is shown replotted in log K_I vs log $\dot{a}$ coordinates in Fig. 2. Also shown in this figure are data under 500 psig hydrostatic pressure from Ref. 18, indicating the change in behavior produced by hydrostatic pressure. Although some nonlinearity of the curve can be seen in log-log coordinates, at least the main trend of the behavior is described by the straight line expressed by

$$\dot{a} = 1 \times 10^{-12} K_I{}^6 \qquad \text{in./min} \tag{8}$$

The fit of the equation in the original semilog coordinates of Ref. 18 is shown in Fig. 3.

Since the change in Poisson's ratio with time is negligible compared with the change in modulus, the approximate relationship between material properties is

$$C_v(t) = 4(1-\nu^2) D_{\mathrm{crp}}(t), \quad D_{\mathrm{crp}}(t) \approx (1/E_I) t^{-\beta}$$

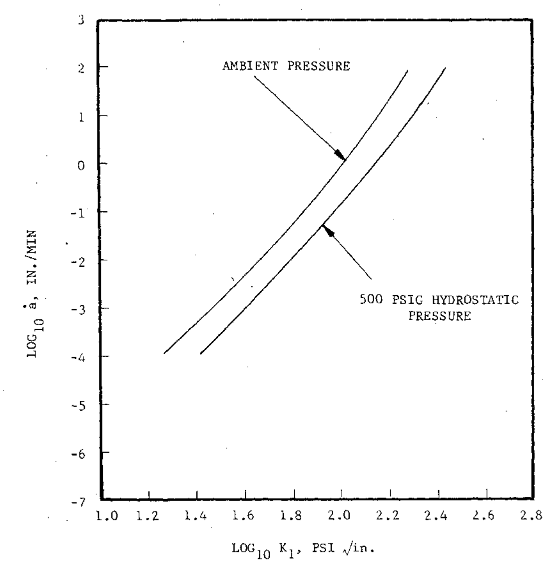

Fig. 2 Relationship between stress intensity factor K_I and crack velocity in PBAN propellant from Ref. 18.

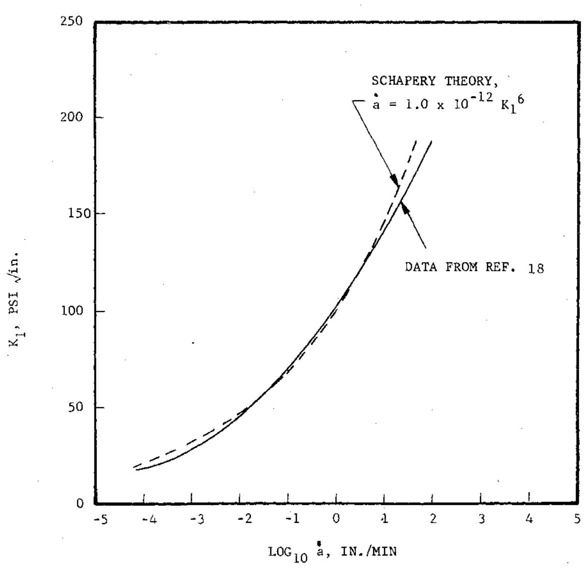

Fig. 3 Comparison of crack velocity theory with PBAN propellant data from Ref. 18.

and, furthermore, n in Shapery's propagation law is given by $n = -\beta$. Thus the exponent of Eq. (4) is given by

$$2[1+(1/0.089)] = 24.5,$$

which clearly is different from the experimental value of 6. It thus seems clear that the fracture properties must be allowed to be crack-velocity dependent.

If the fracture energy for propagation is taken as

$$\Gamma = \Gamma_p \dot{a}^{\ell} \tag{9}$$

a substitution into either Eq. (1) or (2) gives

$$\dot{a} = \left[\frac{C_1 \lambda_n \Pi^n}{\Gamma_p \sigma_m{}^{2n} I_1{}^{2n} 2^{3+n}}\right]^{1/(n+\ell)} K_1 \frac{2(1+n)}{n+\ell} \tag{10}$$

and, if ℓ is taken as 0.274 (with $n = 0.089$), there results a value of 6 for the exponent. Thus if the fracture energy for propagation is taken as

$$\Gamma = \Gamma_p \dot{a}^{0.274} \tag{11}$$

and the combination of material properties in brackets is taken at the appropriate value, the crack propagation law is given by

$$\dot{a} = 1\times 10^{-12} \quad K_1{}^6 \qquad \text{in./min}$$

(where K_1 is in psi in.$^{1/2}$), thus fitting the data of Ref. 18 given previously.

Bennett[19] has measured the stress intensity factors at initiation of crack spreading for the PBAN propellant, using constant strain rate tests on centrally cracked biaxial strip specimens. The results are presented in Ref. 19 in the form of γ as a function of time by using the data-reduction procedure

$$\gamma(t) = \frac{K_1{}^2(t)}{2E_{\text{rel}}(t)}$$

where t is the time to failure in the constant strain rate tests. The reverse procedure has been employed here to calculate K_1 for initiation vs time to failure, which is shown plotted in Fig. 4.

Schapery's initiation law, Eq. (5), easily can be put in the form of the preceding data. In constant strain rate tests, the applied stress is related to time by

$$\sigma(t) = R_\epsilon \int_0^t E_{\text{rel}}(t-\tau)\,d\tau = \frac{R_\epsilon E_1 t^{1+\beta}}{1+\beta} \tag{12}$$

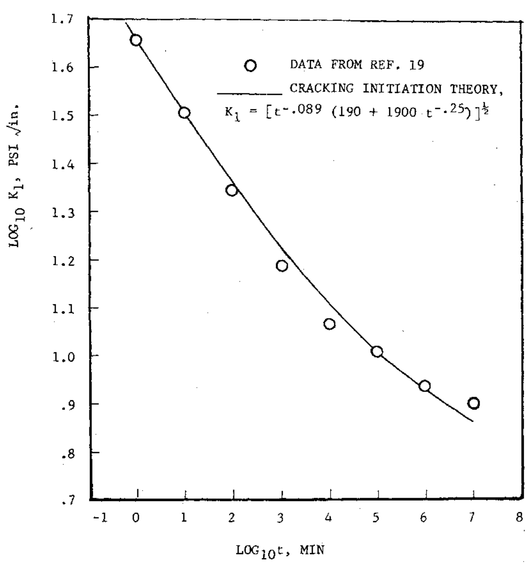

Fig. 4 Comparison of cracking initiation theory with data for PBAN propellant from Ref. 19.

and thus

$$K_1{}^2(t) = \frac{\Pi a R_\epsilon{}^2 E_1{}^2 t^{2(1+\beta)}}{(1+\beta)^2} \tag{13}$$

The term

$$C_v{}^{(2)} = \frac{2(1+\beta)C_1}{\tau^{2(1+\beta)}} \int_0^t (t-\tau)^{-\beta} \epsilon^{(1+2\beta)}\, d\tau \tag{14}$$

It is convenient here to use the binomial expansion

$$(t-\tau)^m = t^m - \frac{m t^{m-1}}{1!}\tau + \frac{m(m-1)t^{m-2}}{2!}\tau^2 - \frac{m(m-1)(m-2)t^{m-3}\tau^3}{3!} + \dots \tag{15}$$

so that there results

$$C_v{}^{(2)} = 2(1+\beta)\delta C_1 t^{-\beta} = 2(1+\beta)\delta C_1 t^n \tag{16}$$

and δ is a numerical factor given by

$$\delta = \frac{1}{2\beta+2} + \frac{\beta}{1!(2\beta+3)} + \frac{\beta(1+\beta)}{2!(2\beta+4)} + \frac{\beta(1+\beta)(2+\beta)}{3!(2\beta+5)} + \dots \tag{17}$$

and finally

$$K_1(t_i) = \left[\frac{4\Gamma_i t_i{}^{\beta}}{\delta(1+\beta)C_1}\right]^{1/2} \tag{18}$$

Thus if Γ_i, the fracture energy for cracking initiation, is constant, the log-log plot of Fig. 4 should have a slope equal to $\beta/2$ and thus $-0.089/2$. This slope is totally at variance with the experimental results of Fig. 4. It is concluded that Γ_i the fracture energy for initiation, must be history dependent in some manner. The choice is somewhat arbitrary at this point, but an expression for Γ of the form

$$\Gamma_i = g_1 + g_2 t^{s'} \tag{19}$$

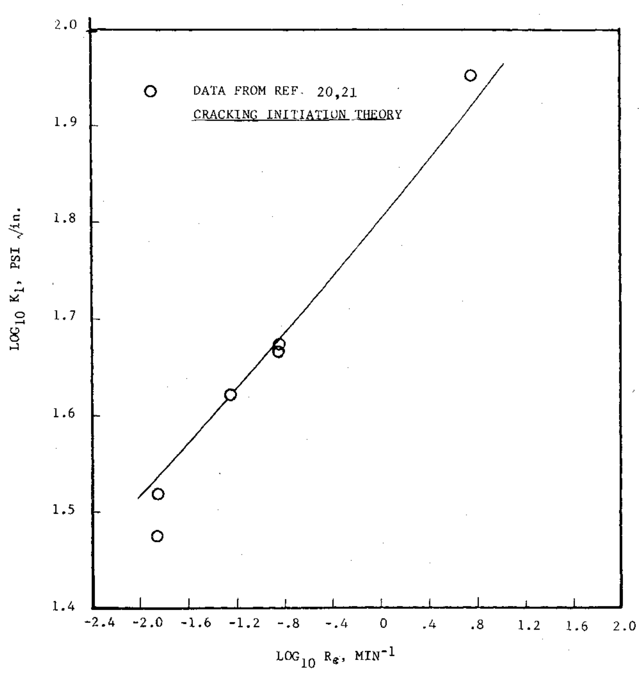

Fig. 5 Comparison of cracking initiation theory with data for PBAN propellant from Refs. 20 and 21.

was chosen, This choice will be discussed further subsequently. Fitting this expression to the data results in

$$\frac{4\Gamma}{(1+\beta)\,\delta C_I} = 190 + 1900\, t^{-0.25} \tag{20}$$

and thus

$$K_I(t_i) = [t^{-0.089}(190 + 1900\, t^{-0.25})]^{1/2} \text{psi in.}^{1/2} \tag{21}$$

(where t is in minutes) which also is shown in Fig. 4. The closeness of the agreement, of course, merely reflects on the accuracy of the curve fit.

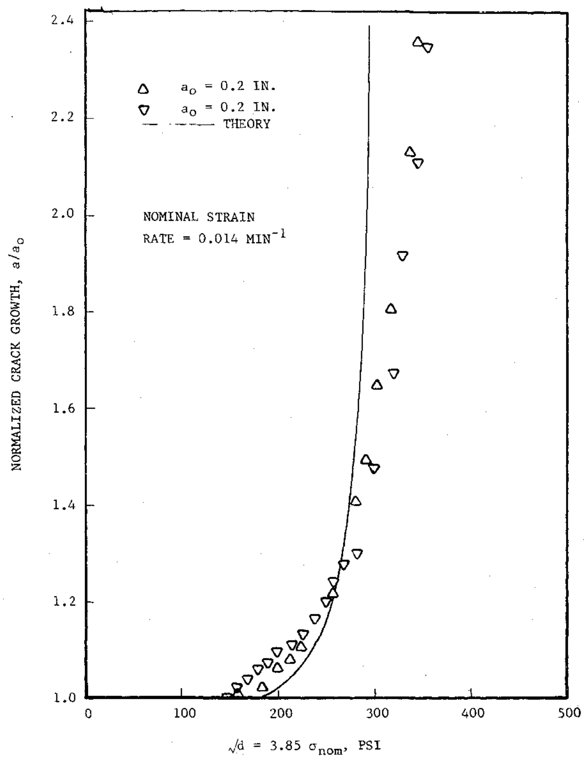

Fig. 6 Comparison of crack growth theory with PBAN propellant data from Ref. 20.

Further experimental data are available for the PBAN propellant in the work of Jacobs et al.[20,21] for constant load and constant strain rate tests of centrally cracked sheets. The results for the constant strain rate tests are given in terms of a damage factor $d^{1/2}$, which is numerically equal to 3.85 times the applied nominal stress for all of the tests. The stress intensity factors thus can be calculated from the published results. The K_I values thus calculated are shown as a function of the applied nominal strain rate in Fig. 5.

The preceding initiation condition, Eqs. (5) and (20), can be solved implicitly for K_I as a function of strain rate by substituting Eq. (12), relating stress and time, into Eqs. (5) and (20). The resulting K_I values at cracking initiation are shown in Fig. 5 as a function of strain rate. It can be seen that good agreement with the experimental data is obtained.

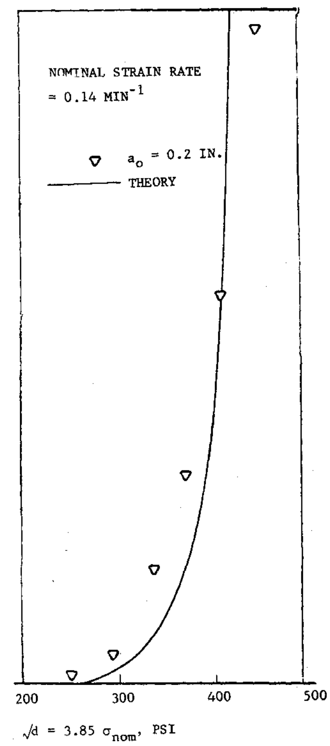

Fig. 8 Comparison of crack growth theory with PBAN propellant data from Ref. 20.

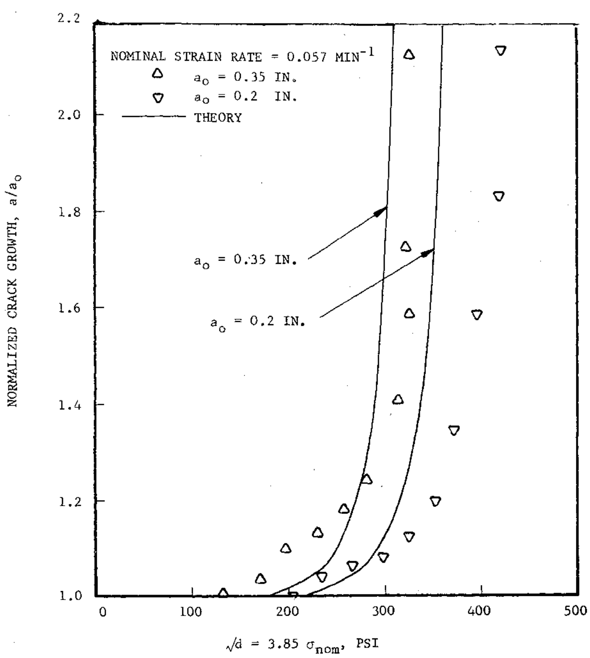

Fig. 7 Comparison of crack growth theory with PBAN propellant data from Ref. 20.

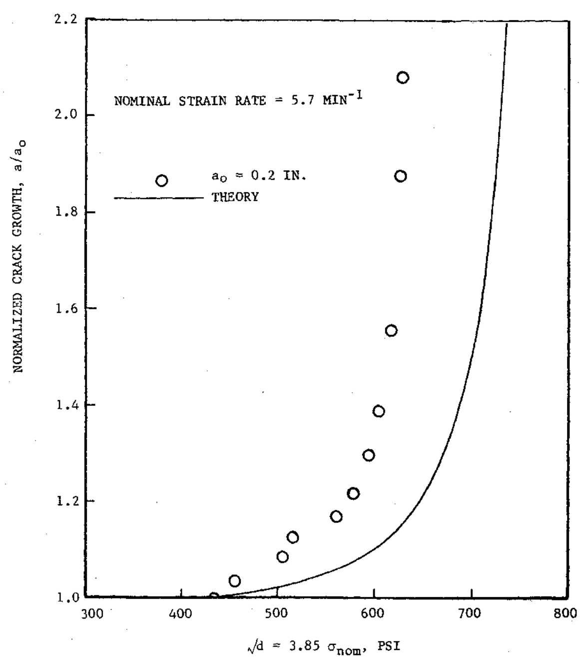

Fig. 9 Comparison of crack growth with PBAN propellant data from Ref. 20.

The crack propagation observed by Jacobs et al.[20] has been reported for the constant strain rate tests in the form of crack length a divided by original crack length a_0 as a function of the nominal applied stress σ (expressed as $d^{1/2} = 3.850\sigma$). The theoretical prediction can be put in this form by integrating Eq. (10), using the chain rule to change the independent variable from time to stress, and using the relationship between time and stress, Eq. (12), to get

$$a/a^0 = [1 - C_4 a_0 (1/r)(\sigma^p - \sigma_c{}^p)]^{-r} \tag{22}$$

where

$$C_4 = \frac{A}{r}\frac{\Pi^{q/2}}{(1-n)}\left(\frac{1-n}{R_\epsilon E_1}\right)^{1/(1-n)}\frac{1}{p} \tag{23}$$

$$p = q + \frac{1}{(1-n)}, \qquad q = \frac{2(1+n)}{n+\ell}, \qquad r = \frac{1}{q/2-1}$$

Substituting the previous values ($A = 1\times10^{-12}$ psi^{-q}min^{-1} in.$^{1-q/2}$, $q=6$, $n=0.089$, and $E_1 = 531$ psi) gives the results shown in Figs. 6-9 compared with the experimental values. It should be noted that the expression for K_I has been taken simply as $k_I = \sigma(\Pi a)^{1/2}$, and a finite width specimen correction has not been made. A value of $a/b = 0.2$, where b is a sheet half-width, gives a finite width correction of 2½%.[22] Since the original propellant sheets were 3.5 in. square, the width correction can be ignored for crack extensions up to $a/a_0 = 1.75$ for the cracks with $a_0 = 0.2$; however, the crack with $a_0 = 0.35$ in. is in error by 7 or 8% at the same crack extension ratio. It also should be noted that some of the experimental results reported in Ref. 20 were for tests labeled biaxial. This refers to tests in which a lateral prestrain was applied and then held constant.[23] Since the strain level was not reported, calculations could not be made for those tests. The K_I expression is, of course, unchanged by the lateral stress, but the stress-time relationship is changed by the biaxial restraint.

The final series of tests to be considered is constant load tests on cracked specimens.[20] These results are reported in terms of crack extension a/a_0 as a function of time after initial crack extension. The theory can be put in this form by integrating Eq. (10) directly to get

$$a/a_0 = [1 - (A\Pi^{q/2}/r) a_0 (1/r)\, \sigma^q (t - t_i)]^{-r} \tag{24}$$

Using the previous values for the constants gives the results shown compared with the experimental results in Fig. 10. In these tests, particularly, the experimental results are quite scattered, and the agreement with the theory is also not good. Note that the effect of stress level does not seem to be consistent in the experimental results. The experimental crack velocities also are not in agreement with the data shown in Fig. 3; however, the range of data scatter for the latter figure was not reported, and thus it is difficult to resolve this inconsistency.

Discussion

The striking aspect of the Schapery crack propagation theory under consideration is the good agreement between theory and experiment over a large range of variables. It is important to keep this in mind, since at the very least it appears that a practical fracture theory has been developed by Schapery. It appears to this author that the assumptions of the underlying Barenblatt model are judged best in terms of the predictive value thus obtained, and Schapery's generalization to viscoelastic media is subject to the same experimental qualification. Schapery's model places significant restrictions on the possible crack propagation behavior. The chief restriction is expressed by Eq. (3), which states that the crack velocity depends on the current value of the stress intensity factor, rather than on a more general history dependence. Schapery also obtains the result that the fracture energy for propagation depends at most on the instantaneous crack velocity, but not higher derivatives. Thus it is satisfying to see such good agreement between theory and experiment.

The key to the success of the present application of Schapery's theory to a PBAN solid propellant lies in the use of a time-dependent fracture energy for the initiation of crack propagation, and a crack-velocity-dependent fracture energy for crack propagation. These two energies are essentially the same, as can be seen in the following argument. The crack velocity is related to a characteristic time for crack propagation by Schapery[15,16] as

$$t_\alpha = \lambda n^{(1/n)} \alpha/\dot{a} \approx \alpha/\dot{a}$$

where α is the length of the cohesive zone at the crack tip, and $\lambda_n{}^{1/n}$ is a parameter approximately equal to one-third. Thus the dependence of crack propagation fracture energy on crack velocity can be considered also to be a dependence on the loading time as

$$\Gamma_{\text{prop}} = \Gamma_p \dot{a}^{0.274} \approx \alpha \Gamma_p t_\alpha - 0.274$$

Now the size of the cohesive zone is on the order of 10^{-3} in. or less[5,24] for relatively homogeneous polymeric materials. Although not well determined, the cohesive zone may be larger than this in the present case because of the coarse microstructure of the filled solid propellants. In any case, the characteristic times for the crack velocities of Fig. 3 range from say 10^{-5} to 10 min. This is on the short time scale of, or

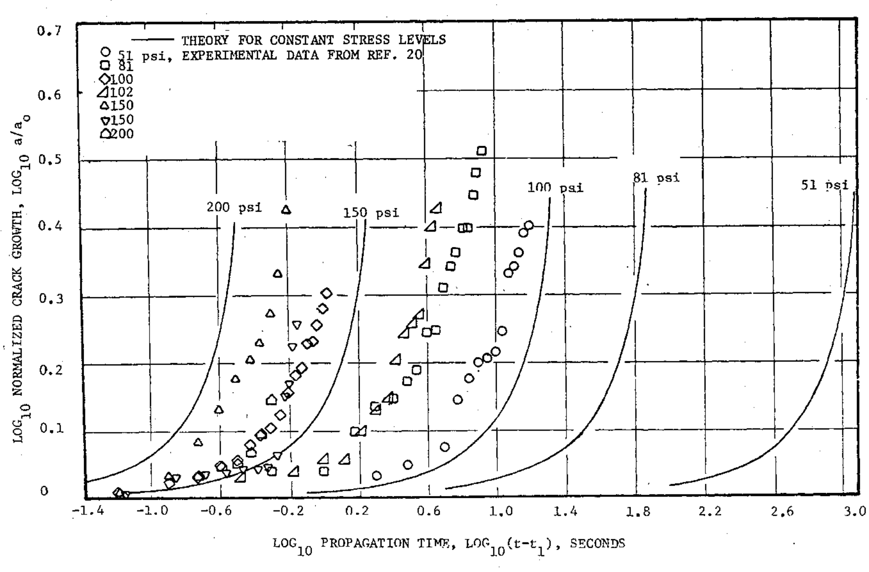

Fig. 10 Comparison of crack growth theory for constant stress tests with PBAN propellant data from Ref. 20.

outside the range of, the times used for characterizing the cracking initiation fracture energy, which was given by (see Fig. 4)

$$\Gamma_{init} = \text{const}\ (190 + 1900\ t^{-0.25})$$

This expression obviously behaves as $t^{-0.25}$ for short loading times. Thus the propagation and initiation fracture energies have time behaviors of $t^{-0.274}$ and $t^{-0.25}$, respectively. If this difference is judged not significant, then it seems that the initiation and propagation fracture energies have the same form. The propagation energy appears as a multiple of a group of fracture properties and thus has not been evaluated by itself. The constants multiplying the initiation energy can be evaluated in terms of previously given parameters so that

$$\Gamma_{init} = 0.12 + 1.20\ t^{-0.25}\ \text{in.-lb/in.}^2$$

This fracture initiation energy may have a much more complex history dependence than indicated previously. The preceding form has been developed from a single type of loading, nominal constant strain rate, and other loading histories could give different forms which still could provide the crack velocity dependence required. However, Swanson[25] has shown previously that time to crack initiation is a relatively history-insensitive parameter, at least for one kind of propellant and a restricted class of rather smooth loadings. It should be pointed out, however, that the simple time dependence given previously probably will not apply to cyclic loadings or other loadings that deviate strongly from constant strain rate. Clearly more experimental evidence is required here.

The present result clearly is different from that observed by Schapery,[16] using data of Knauss[5] on Solithane 113, an unfilled polyurethane rubber. Schapery and Knauss both obtained a good fit to the Solithane data by using a constant fracture energy. In contrast, Bennett et al.[26] observed strong time dependence of fracture energy in cracking initiation tests on a polybutadiene acrylic acid rubber. Thus, although it is tempting to ascribe the present result of a time-dependent fracture energy to the high solids loading of solid propellants, this is not borne out by the work of Bennett et al. It should be noted that in the present study the time dependence of the fracture energy for crack propagation could have been replaced by a dependence of the cohesion zone length on crack velocity. Identical results would have been obtained, and the choice is thus arbitrary. In view of the consistency of the crack initation and propagation energies thus obtained, it appears that the approach taken in the present work has merit.

Finally, it should be pointed out that the predictions of Schapery's theory are not greatly different from those of previous theories in certain respects. For example, as shown by Bennett[19] and Bennett et al.[26] the Williams flaw analogy[4] can be interpreted as yielding a criterion for the initiation of cracking of the form

$$K_I^{\,2}(t) \approx 2E_{rel}(t)\gamma$$

which is similar to Eq. (5). Also, Knauss[5] has developed an expression for crack velocity which depends on the stress intensity factor raised to a power that is equivalent in some cases to Eq. (4).

Summary and Conclusions

Schapery's viscoelastic generalization of the Barenblatt fracture model has been evaluated in terms of published experimental data for a PBAN solid propellant. Cracking behavior has been reported for this material by Francis et al., Bennett, and Jacobs et al. The theory predicts both the initiation and subsequent velocity of crack propagation in linear viscoelastic materials. Comparison of theory and experiment over a wide range of experimental variables is very good, indicating that the theory provides a useful tool for handling cracking-related structural problems.

The agreement between theory and experiment requires the assumption of a time-dependent fracture energy that appears in both crack velocity and cracking initiation in an equivalent form. This time-dependence may be a special form of a more general (and unknown) history dependence of fracture energy.

References

[1] Williams, M.L., "The Fracture of Viscoelastic Material," *Fracture of Solids*, edited by D. C. Drucker and J. J. Gilman, Interscience, New York, 1963.

[2] Bueche, F. and Halpin J. C., "Molecular Theory for the Tensile Strength of Gum Elastomers," *Journal of Applied Physics*, Vol. 35, Jan. 1970, pp. 497-507.

[3] Rivlin, R. S. and Thomas, A. G., "Rupture of Rubber, I, Characteristic Energy for Tearing," *Journal of Polymer Science*, Vol. 10, March 1953, pp. 291-318.

[4] Williams, M. L., "Initiation and Growth of Viscoelastic Fracture," *International Journal of Fracture Mechanics*, Vol. 1, Dec. 1964, pp. 292-310.

[5] Knauss, W. G., "Delayed Failure – The Griffith Problem for Linearly Viscoelastic Materials," *International Journal of Fracture Mechanics*, Vol. 6, Mar. 1970, pp. 7-20.

[6] Mueller, H. K. and Knauss, W. G., "Crack Propagation in a Linearly Viscoelastic Strip," *Journal of Applied Mechanics*, Vol. 38, June 1971, pp. 483-488.

[7] Knauss, W. G. and Dietmann, H., "Crack Propagation Under Variable Load Histories in Linearly Viscoelastic Solids," *International Journal of Engineering Science*, Vol. 8, Aug. 1970, pp. 643-656.

[8] Knauss, W. G., "Stable and Unstable Crack Growth in Viscoelastic Media," *Transactions of the Society of Rheology*, Vol. 13, 1969, pp. 291-313.

[9] Cherepanov. G. P., "Crack Propagation in Continuous Media," *Journal of Applied Mathematics and Mechanics (PMM)*, Vol. 31, 1967, pp. 503-512.

[10] Cherepanov, G. P., "Cracks in Solids," *International Journal of Solids and Structures*, Vol. 4, 1968, pp. 811-831.

[11] Graham, G. A. C., "Two Extending Crack Problems in Linear Viscoelasticity Theory," *Quarterly of Applied Mathematics*, Vol. 27, June 1970, pp. 497-507.

[12] Nuismer, R. J., "On the Governing Equation for Quasi-Static Crack Growth in Linearly Viscoelastic Materials," *Journal of Applied Mechanics*, Vol. 41, Sept. 1970, pp. 631-634.

[13] Knauss, W. G. and Mueller, H. K., Discussion of Reference 12, "On the Governing Equation for Quasi-Static Crack Growth in Linearly Viscoelastic Materials," *Journal of Applied Mechanics*, Vol. 42, June 1975, pp. 520-521.

[14] Knauss, W. G., "The Mechanics of Polymer Fracture," *Applied Mechanics Reviews*, Vol. 26, June 1973, pp. 1-17.

[15] Schapery, R. A., "A Theory of Crack Initiation and Growth in Viscoelastic Media, I. Theoretical Development," *International Journal of Fracture*, Vol. 11, Feb. 1975, pp. 141-159.

[16] Schapery, R. A., "A Theory of Crack Growth in Viscoelastic Media," Texas A&M Univ. Rept. MM 2764-73-1, 1973.

[17] Barenblatt, G. I., "The Mathematical Theory of Equilibrium Cracks in Brittle Fracture *Advances in Applied Mechanics*, Vol. VII, Academic Press, New York, 1962, pp. 55-129

[18] Francis, E. C., Carlton, C. H., and Lindsey, G. H., "Viscoelastic Fracture of Solid Propellants in Pressurization Loading Conditions," *Journal of Spacecraft and Rockets*, Vol. 11, Oct. 1974, pp. 691-696.

[19] Bennett, S. J., "The Use of Energy Balance in Rocket Motor Grain Integrity Studies," *JANNAF Mechanical Behavior Working Group, 8th Meeting*, CPIA Publ. 193, Vol. 1, 1970, pp. 393-403.

[20] Jacobs, H. R., Hufferd, W. L., and Williams, M. L., "Further Studies of the Critical Nature of Cracks in Solid Propellant Grains," Univ. of Utah, Salt Lake City, Utah, Rept. AFRPL-TR-75-14, 1975.

[21] Hufferd, W. L., Laheru, K. L., and Jacobs, H. R., "On the Fracture of Viscoelastic Materials," *1975 Annual JANNAF Operational Serviceability Working Group and Structures and Mechanical Behavior Working Group Meeting*, Feb. 25-28, 1975.

[22] Rooke, D. P., "Width Corrections in Fracture Mechanics," *Engineering Fracture Mechanics*, Vol. 1, Apr. 1970, pp. 727-728.

[23] Hufferd, W. L., personal communication, Univ. of Utah, Salt Lake City, Utah, May 1975.

[24] Constable, I., Culver, L. E., and Williams, J. G., "Notch Root Radii Effects in the Fatigue of Polymers," *International Journal of Fracture Mechanics*, Vol. 6, Sept. 1970, pp. 279-285.

[25] Swanson, S. R., "Crack Propagation in Double-Base Propellants Under Ignition Loading," *7th Meeting ICRPS Mechanical Behavior Working Group, CPIA Publ. 177, 1968*, pp. 131-142.

[26] Bennett, S. J., Anderson, G. P., and Williams, M. L., "The Time Dependence of Surface Energy in Cohesive Fracture," *Journal of Applied Polymer Science*, Vol. 14, Mar. 1970, pp. 735-745.